120 Midterm Solutions

Q1 (a) G acts on the set of vertices. If $g \in G$ fixes $(1,0,0)$, it also fixes $(-1,0,0)$, so is a rotation about the x-axis. There are four such elements in the stabaliser of $(1,0,0)$.

Given any two vertices v, w of the octahedron, there exists a co-ordinate axis perpendicular to v, and w. A rotation about this coordinate axis will give $g \in G$ with $g \cdot v = w$. Hence G acts on the set of vertices with a single orbit.

By orbit - stabaliser: $|G| = 4 \times 6 = 24$

(b). Solution #1: The action of G on the set of vertices realises G as a subgroup of S_6. In this way we can talk about the sign of an element of G.

The set of even elements forms a normal subgroup of G of index 1 or 2. (it is the kernel of $\text{sgn}: G \to \{\pm 1\}$).

Rotation by 90° about a coordinate axis is a 4-cycle, hence odd. Thus the subgroup of even elements must have index 2, so G is not simple.

Solution #2. Let n_2 be the number of Sylow 2-subgroups of G. By the third Sylow theorem, $n_2 \mid 3$. So $n_2 = 1$ or $n_2 = 3$.

If $n_2 = 1$, then G has a unique Sylow 2-subgroup, which is normal.

If $n_2 = 3$, consider the action of G on the set of Sylow 2-subgroups of G by conjugation.

This defines a homomorphism $\varphi: G \longrightarrow S_3$, which is nontrivial by 2nd Sylow theorem.

Since $|S_3| = 6 < 24 = |G|$, $|\ker \varphi| \neq 1$. φ is nontrivial so $\ker \varphi \neq G$.

$\therefore \ker \varphi$ is a nontrivial normal subgroup, so G is not simple.

Solution #3. As in solution #2, it suffices to find a set of 3 elements on which G acts nontrivially. Take the set of coordinate axes.

Q2: Let G be a 3-transitive subgroup which contains a 3-cycle.

$\therefore$ $(a\,b\,c) \in G$ for some $a, b, c \in \{1, 2, \dots, n\}$

Let $(x\,y\,z)$ be a 3-cycle in S_n.

Since G is 3-transitive, $\exists\ \sigma \in G$ with $\sigma(a) = x$, $\sigma(b) = y$, $\sigma(c) = z$.

$$\therefore (x\,y\,z) = \sigma \cdot (a\,b\,c) \cdot \sigma^{-1} \in G.$$

By a homework question, A_n is generated by 3-cycles.

$\therefore$ A_n is a subgroup of G.

Since A_n has index 2 in G, the only options are $G = A_n$ or $G = S_n$.

Q3 (a)

$$\begin{pmatrix} 1 & 1 & \cdot \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & \cdot & 1 \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdot & 1 \\ & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix}.$$

The group generated by $\begin{pmatrix} 1 & 1 & \\ \cdot & 1 & 1 \\ & & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$ contains the five elements $\begin{pmatrix} 1 & & \cdot \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ \cdot & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & \cdot & 1 \\ \cdot & 1 & \\ \cdot & & 1 \end{pmatrix}, \begin{pmatrix} 1 & \cdot & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$.

This group is a subgroup of U, and $|U| = 8$.

By Lagrange's theorem, the order of this subgroup divides 8. Since it is at least 5, this order must be 8, proving that U is generated by $\begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$.

(b).

$$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ So $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is a single orbit.

$$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ So $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an orbit.

$$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}$$ $a \in \{0,1\}$. So $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an orbit

$$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ c \\ 1 \end{pmatrix}$$ $b, c \in \{0,1\}$. So $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an orbit.

(c). Consider the orbit with four elements from (b) which we draw in the following way:

$$\begin{tikzcd}
\begin{pmatrix}0\\0\\1\end{pmatrix} \arrow[d, bend right] \arrow[d, dashed, shift right] & \begin{pmatrix}1\\1\\1\end{pmatrix} \arrow[l, bend right] \arrow[d, dashed, shift right] \\
\begin{pmatrix}0\\1\\1\end{pmatrix} \arrow[r, bend right] \arrow[u, dashed, shift right] & \begin{pmatrix}1\\0\\1\end{pmatrix} \arrow[u, bend right] \arrow[u, dashed, shift right]
\end{tikzcd}$$

Solid lines = action of $\begin{pmatrix}1&1&\cdot\\\cdot&1&1\\\cdot&\cdot&1\end{pmatrix}$.

Dashed lines = action of $\begin{pmatrix}1&\cdot&\cdot\\\cdot&1&1\\\cdot&\cdot&1\end{pmatrix}$.

Thinking of these four vectors as vertices of a square, the action of $\begin{pmatrix}1&1&\cdot\\\cdot&1&1\\\cdot&\cdot&1\end{pmatrix}$ and $\begin{pmatrix}1&\cdot&\cdot\\\cdot&1&1\\\cdot&\cdot&1\end{pmatrix}$ is by a rotation and a reflection respectively, which are known to generate the dihedral group D_4.
This allows us to conclude that D_4 is a quotient of $\mathcal{U}$ via a homomorphism $\begin{pmatrix}1&1&\cdot\\\cdot&1&1\\\cdot&\cdot&1\end{pmatrix} \longmapsto \text{rot}$, $\begin{pmatrix}1&\cdot&\cdot\\\cdot&1&1\\\cdot&\cdot&1\end{pmatrix} \longmapsto \text{refl}$.
Since $|\mathcal{U}| = |D_4| = 8$, this homomorphism is an isomorphism.

(d) We know that $\mathcal{U}$ is a Sylow 2-subgroup of $GL_3(\mathbb{F}_2)$. By the 2nd Sylow Theorem, any two Sylow 2-subgroups are conjugate.
So P is conjugate to $\mathcal{U}$.
In particular $P \cong \mathcal{U} \cong D_4$.
Let $P = g\mathcal{U}g^{-1}$ and O be a $\mathcal{U}$-orbit.
Let $x \in gO$ and $p \in P$. Then $x = gy$, $p = gug^{-1}$ $y \in O$, $u \in \mathcal{U}$.

$$p\,x = gug^{-1} \cdot gy = g(uy) \in gO.$$

So gO is an orbit of P. $|gO| = |O|$.
$\therefore$ The P-orbits on $\mathbb{F}_2^3$ have sizes $1, 1, 2, 4$.

Q4: To prove that a and b generate a free group we need to show that if $a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k} = 1$ then

$m_1 = m_2 = \cdots = m_k = n_1 = n_2 = \cdots = n_k = 0$. (here $m_i, n_i \in \mathbb{Z}$).

Consider an expression $g = a_1^{m_1} b_1^{n_1} \cdots a^{m_k} b^{n_k}$

Without loss of generality, we may assume $n_i \neq 0$ for all i and $m_i \neq 0$ for $i \geq 2$

Pick $x \in X \setminus (A^+ \cup A^- \cup B^+ \cup B^-)$.

$$x \in (X \setminus B^+) \cap (X \setminus B^-)$$

$$\therefore \quad b^{n_k} x \in B^+ \cup B^- \qquad (*)$$

$$\therefore \quad b^{n_k} x \in (X \setminus A^-) \cap (X \setminus A^+)$$

$$\therefore \quad a^{m_k} b^{n_k} x \in A^+ \cup A^-$$

$$\vdots$$

$\therefore$ $gx \in A^+ \cup A^- \cup B^+ \cup B^-$ i.e. $gx \neq x$

$\therefore$ $g \neq 1$ as required.

$(*)$. It would be useful first to prove that.

Lemma: If $n \neq 0$,

$$b^n\big((X \setminus B^+) \cap (X \setminus B^-)\big) \subset B^+ \cup B^-.$$