1. Stirling's Formula via Hayman's Method

Throughout we write $f(n) \sim g(n)$ to mean that $\lim_{n\to\infty} f(n)/g(n) = 1$.

The purpose is to prove Stirling's Formula

Theorem 1.1 (Stirling's Formula)**.**

$$n! \sim \sqrt{2\pi n}(\frac{n}{e})^n.$$

The function e^z/z^{n+1} is meromorphic with a single pole at 0, where the residue is $1/n!$. Therefore

$$\frac{1}{n!} = \frac{1}{2\pi i}\int_C \frac{e^z}{z^{n+1}}dz$$

where C is some circle centred at the origin, say of radius r.

We parametrise C by $z = re^{i\theta}$. After some simplification, this leads us to

$$\frac{1}{n!} = \frac{1}{2\pi r^n}\int_{-\pi}^{\pi} \exp(re^{i\theta} - in\theta)d\theta.$$

We will approximate the integral over $[-\pi, \pi]$ by an integral over $[-\epsilon, \epsilon]$. The justification for this step will be delayed until later. Note that we have the freedom to choose r and ϵ as functions of n.

To choose r, we force the coefficient of θ in the power series expansion of the exponent $re^{i\theta} - in\theta$ to be zero. This forces $r = n$. It will turn out that $\epsilon = n^{-2/5}$ will work as a choice of ϵ.

We approximate the exponent of the integrand $ne^{i\theta} - in\theta$ by its degree two Taylor approximation $n - n\theta^2/2$.

We need to compare the two integrals

$$I_1 = \int_{-\epsilon}^{\epsilon} \exp(ne^{i\theta} - in\theta)d\theta \quad \text{and} \quad I_2 = \int_{-\epsilon}^{\epsilon} \exp(n - n\theta^2/2)d\theta$$

There exists a constant C_1 such that $|(ne^{i\theta} - in\theta) - (n - n\theta^2/2)| \leq C_1 n|\theta|^3$ for all $|\theta| \leq \pi$ (Proof: the function $\theta \to |(e^{i\theta} - i\theta) - (1 - \theta^2/2)/\theta^3|$ is continuous on the compact set $|\theta| \leq \pi$). A similar argument shows there exists a constant C_2 such that

$$|\exp((ne^{i\theta} - in\theta) - (n - n\theta^2/2)) - 1| \leq C_2 n|\theta|^3$$

whenever $n|\theta|^3 \leq 1$. Let us now suppose that $n\epsilon^3 \to 0$ as $n \to \infty$. Then for n sufficiently large

$$\begin{aligned}
|I_1 - I_2| &\leq \int_{-\epsilon}^{\epsilon} |\exp((ne^{i\theta} - in\theta) - (n - n\theta^2/2)) - 1| e^{n - n\theta^2/2}d\theta \\
&\leq \int_{\epsilon}^{\epsilon} C_2 n\epsilon^3 e^{n - n\theta^2/2}d\theta \\
&= C_2 n\epsilon^3 I_2.
\end{aligned}$$

Since $n\epsilon^3$ tends to zero as n tends to infinity we obtain

$$\lim_{n\to\infty} \frac{I_1}{I_2} = 1.$$

Making the change of variable $y = \sqrt{n/2}\theta$,

$$I_2 = e^n \int_{-\epsilon\sqrt{n/2}}^{\epsilon\sqrt{n/2}} \sqrt{2/n} e^{-y^2} dy$$

Now we assume that $n\epsilon^2 \to \infty$ as $n \to \infty$ and obtain

$$\lim_{n\to\infty} I_2 e^{-n}\sqrt{2/n} = \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}.$$

Hence we have the asymptotic formula

$$I_1 \sim \sqrt{2\pi} e^n/\sqrt{n}.$$

It remains to justify the passage from an integral over $[-\pi, \pi]$ to an integral over $[-\epsilon, \epsilon]$. To do this, we estimate

$$\left|\int_{\epsilon}^{\pi} \exp(ne^{i\theta} - in\theta)d\theta\right| \le (\pi - \epsilon)e^{n\cos\epsilon}$$

(The second factor is the greatest absolute value obtained by the integrand) We need an (again, easily and similarly obtained) estimate of the form $1 - \cos\epsilon \ge C_3\epsilon^2$ for $\epsilon \le \pi$ and therefore this integral is bounded above by $e^n e^{-C_3 n\epsilon^2}$.

If, say, $\epsilon = n^{-2/5}$, then it is now easy to see that

$$\lim_{n\to\infty} \frac{e^n e^{-C_3 n\epsilon^2}}{\sqrt{2\pi}e^n/\sqrt{n}} = 0$$

and similarly we may deal with the integral from $-\pi$ to $-\epsilon$.

Therefore

$$\int_{-\pi}^{\pi} \exp(ne^{i\theta} - in\theta)d\theta \sim I_1$$

which completes the proof.