Homework 7

Due: Thursday May 30, 2013

1. Find all possible values of i^i.

2. Let a and c be positive real numbers. Prove that

$$\lim_{N\to\infty}\int_{c-iN}^{c+iN}\frac{a^s}{s}ds=\begin{cases}2\pi i & \text{if } a>1\\ 0 & \text{if } a<1\end{cases}$$

   where the integral is taken over a vertical line segment.

3. Let $0<a<b$ be real numbers. Prove that there exists $\delta>0$ such that $\zeta(s)\neq 0$ for all $s=\sigma+it$ with $\sigma>1-\delta$ and $a<t<b$.

4. Let c and x be real numbers greater than 1. Find an exact formula for

$$\int_{c-i\infty}^{c+i\infty}\frac{x^s\zeta(s)}{s(s+1)}ds$$

   and compute the residue of the integrand at the point $s=1$.

5. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. Define

$$L=\limsup_{n\to\infty}\left(1+\frac{\log|a_n|}{\log n}\right).$$

   Prove that the Dirichlet series $f(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ is uniformly convergent on compact subsets of $\{s\in\mathbb{C}\mid \Re(s)>L\}$. (Unlike the case of a power series, this is not a formula for the greatest open half-plane on which the series converges)