Solutions to 116 Homework 6

1. Integrating in the Riemann-Stieltjes sense, we have

$$\begin{aligned}\pi(x) = \sum_{p\le x} \frac{\log p}{\log p} &= \lim_{\epsilon\downarrow 0} \int_{2-\epsilon}^{x+\epsilon} \frac{1}{\log t} d\theta(t) \\ &= \lim_{\epsilon\downarrow 0} \left.\frac{\theta(t)}{\log t}\right|_{2-\epsilon}^{x+\epsilon} - \int_{2-\epsilon}^{x+\epsilon} \theta(t) d\left(\frac{1}{\log t}\right) \\ &= \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2} dt\end{aligned}$$

Or we can evaluate the identity explicitly. Note that

$$\int_a^b \frac{1}{t(\log t)^2} dt = \frac{1}{\log a} - \frac{1}{\log b}$$

and let $p_1 < p_2 < \ldots < p_n$ be the primes $\le x$, so that $\pi(x) = n$. Then

$$\begin{aligned}&\frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2} dt \\ &= \frac{\theta(p_n)}{\log x} + \theta(p_n) \int_{p_n}^x \frac{\theta(t)}{t(\log t)^2} dt + \sum_{k=1}^{n-1} \theta(p_k) \int_{p_k}^{p_{k+1}} \frac{\theta(t)}{t(\log t)^2} dt \\ &= \frac{\theta(p_n)}{\log p_n} + \sum_{k=1}^{n-1} \theta(p_k) \left(\frac{1}{\log p_k} - \frac{1}{\log p_{k+1}}\right) \\ &= \sum_{k=1}^{n-1} \frac{\theta(p_{k+1}) - \theta(p_k)}{\log p_k} + \frac{\theta(p_1)}{\log p_1} \\ &= n\end{aligned}$$

2. Since $z \in \mathfrak{h}$ is never real, $cz + d \neq 0$ on $\mathfrak{h}$, so $\rho_A : \mathfrak{h} \to \mathbb{C}$ is well-defined. Take $z \in \mathfrak{h}$, then we calculate

$$\Im \rho_A(z) = \frac{\det A}{|cz+d|^2} \Im z > 0$$

so $\rho_A(\mathfrak{h}) \subset \mathfrak{h}$.

Let B be the matrix $\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, and write ρ_B for the corresponding Mobius transform. Since $\det B = \det A$, as above ρ_B is a well-defined map $\mathfrak{h} \to \mathfrak{h}$. An easy calculation shows that $\rho_A^{-1} = \rho_B$.

3. Let $|f| > \epsilon > 0$ on $C = \partial D$. By assumption, $f_n \to f$ uniformly on the closed disc $\overline{D}$, so we can find an N such that $|f_n - f| < \epsilon/2$ on C whenever $n > N$. In particular, this means that $|f| > |f_n - f|$. So by Rouche's theorem, f and $f + (f_n - f) = f_n$ have the same number of zeros when $n > N$.

4. Take $\Re s > 1$. For any integer N,

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n^s} &= \lim_{\epsilon \downarrow 0} \int_{1-\epsilon}^{N+\epsilon} \frac{1}{x^s} d\lfloor x \rfloor \\
&= \frac{N}{N^s} - \int_1^N \lfloor x \rfloor \left(\frac{-s}{x^{s+1}} dx \right) \\
&= N^{1-s} + s \int_1^N \frac{x}{x^{s+1}} dx - s \int_1^N \frac{x - \lfloor x \rfloor}{x^{s+1}} dx \\
&= N^{1-s} + \frac{s}{s-1}(1 - N^{1-s}) - s \int_1^N \frac{x - \lfloor x \rfloor}{x^{s+1}} dx
\end{aligned}$$

(again we use Riemann-Stieltjes integral)

Since $|N^{1-s}| = N^{1-\Re s}$, take $N \to \infty$ and the result follows.

The last integral converges absolutely iff $\Re s > 0$, since $|x - \lfloor x \rfloor| \leq 1$ but $|x - \lfloor x \rfloor| \geq 1/2$ for $x \in \mathbb{N} + [1/2, 1]$. So the right hand side defines a meromorphic extension of ζ to $\Re s > 0$.