Homework 2

Due: Thursday April 26, 2012

1. Show that every subgroup of a group can appear as a stabaliser in a group action by considering the action of a group G on cosets of a subgroup H by $g \cdot (g'H) = (gg')H$.

2. Prove that the alternating group A_n is generated by 3-cycles.

3. Let $n \geq 5$ be an integer. Prove that all the 3-cycles lie in a single conjugacy class of A_n.

4. Let p be an odd prime and n be a positive integer. Prove that $1 + p$ is an element of order p^{n-1} in the multiplicative group $(\mathbb{Z}/p^n\mathbb{Z})^\times$. (Part of this statement is that $(1 + p)^d$ is not the identity for any positive $d < p^{n-1}$).

5. Find an example of a group G with the property that any two elements generate a proper subgroup.

6. Let F be a field. Let T be the subgroup of $GL_n(F)$ consisting of invertible diagonal matrices. Let B be the subgroup of $GL_n(F)$ consisting of invertible upper-triangular matrices. Let U be the subgroup of $GL_n(F)$ consisting of upper-triangular matrices with all diagonal entries equal to 1. Prove that B is a semi-direct product of T and U.

7. Consider the symmetric group S_{n-1} as the subgroup of S_n permuting the first $n - 1$ letters. Prove that the set

$$\{S_{n-1}, (n-1, n)S_{n-1}, (n-2, n-1, n)S_{n-1}, \ldots, (1, 2, \ldots, n)S_{n-1}\}$$

is precisely the set of left cosets of S_{n-1} in S_n.