Homework 4

Due: Thursday May 3, 2012

1. An *automorphism* of a group G is an isomorphism from G to G. Prove that the set $\mathrm{Aut}\,(G)$ of automorphisms of G is a group with multiplication given by composition of functions.

   Prove that the map $c : G \rightarrow \mathrm{Aut}\,(G)$ defined by $(c(g))(h) = ghg^{-1}$ is a homomorphism, and determine its kernel.

   Prove that the image of c is a normal subgroup of $\mathrm{Aut}\,(G)$ (this is called the group of *inner automorphisms*).

2. For any field k, $\mathbb{P}^1(k)$ is defined to be the set of one-dimensional subspaces of the k-vector space k^2. Show that $SL_2(\mathbb{F}_4) \cong A_5$ by considering the natural action of $SL_2(\mathbb{F}_4)$ on $\mathbb{P}^1(\mathbb{F}_4)$.

3. A group action is said to be *transitive* if there is only one orbit. Let G be a group acting transitively on a finite set X with at least two elements. Prove that there exists $g \in G$ such that $g \cdot x \neq x$ for all $x \in X$. What happens if G and X are allowed to be infinite?

4. The special orthogonal group SO_n is the group of all $n \times n$ real matrices A with $A^{-1} = A^T$ and $\det(A) = 1$. Describe all the conjugacy classes in SO_3. (It is perhaps more useful to think of orthogonal matrices as matrices A such that $Av \cdot Aw = v \cdot w$ for all vectors $v, w \in \mathbb{R}^n$ where $\cdot$ is the dot product on $\mathbb{R}^n$.)

5. Consider the group with presentation

   $$\langle x, y \mid xy = y^2 x, \ yx = x^2 y \rangle.$$

   Prove that this is the trivial group.

6. Consider a standard Rubik's cube. Only turning two adjacent faces, how many different patterns can be reached? (A computer may help here. In particular the program GAP is easy to use and can work easily with permutation groups).

7. Let G be a group generated by a set S and let H be a subgroup of G. Let R be a set of right coset representatives for H. Assume that $1 \in R$. For each $g \in G$, define $\overline{g} \in R$ by $g \in H\overline{g}$. Prove that the set $\{rs(\overline{rs})^{-1} \mid r \in R, s \in S\}$ generates H.