1. (Nesbitt) Prove that for all positive real numbers a, b, and c, the following inequality holds:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

2. (1996 IMO Shortlist) Let a, b, and c be positive real numbers satisfying $abc = 1$. Prove that:

$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \leq 1$$

3. ((1995 IMO Q2)) Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

4. ((1998 APMO Q3)) Let a, b, c be positive real numbers. Prove that:

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{abc}\right)$$

5. ((2000 IMO Q2)) Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1$$

6. ((2001 IMO Q2)) Prove that for all positive real numbers a, b, c:

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$$

7. Let x, y, z be positive reals satisfying $xyz = 1$. Show that:

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

8. Let x, y, z be positive reals. Show that:

$$\frac{3(x+y)(y+z)(z+x)}{8xyz} \geq (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

9. Let x, y, z be positive reals. Show that:

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right) \geq \frac{9}{4}$$

10. Let k, $a_i > 0$ and $s = \sum_{i=1}^{n} a_i$. Show that:

$$\sum_{i=1}^{n}\left(\frac{a_i}{s-a_i}\right)^k \geq \frac{n}{(n-1)^k}$$

11. Let $r_i \geq 1$ for $i = 1, 2, \ldots, n$. Show that:

$$\sum_{i=1}^{n} \frac{1}{r_i + 1} \geq \frac{n}{\left(\prod_{i=1}^{n} r_i\right)^{1/n} + 1}$$

12. Let a_i be positive reals such that $\sum_{i=1}^{n} a_i < 1$. Show that:

$$\frac{a_1 a_2 \ldots a_n \left(1 - (a_1 + a_2 + \cdots + a_n)\right)}{(a_1 + a_2 + \cdots + a_n) \prod_{i=1}^{n}(1 - a_i)} \leq \frac{1}{n^{n+1}}$$

13. (2001 IMO Shortlist) Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality:

$$\frac{x_1}{1 + x_1^2} + \frac{x_2}{1 + x_1^2 + x_2^2} + \cdots + \frac{x_n}{1 + x_1^2 + \cdots + x_n^2} < n$$

14. (2000 AMO Q3) Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be real numbers such that

(i) $0 < x_1 y_1 < x_2 y_2 < \cdots < x_n y_n$ and

(ii) $x_1 + x_2 + \cdots + x_i \geq y_1 + y_2 + \cdots + y_i$ for $1 \leq i \leq n$.

Prove that:

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \leq \frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}.$$

When does equality occur?

15. All coefficients of a quadratic polynomial $f(x) = ax^2 + bx + c$ are positive, and $a + b + c = 1$. Prove that the inequality

$$f(x_1) f(x_2) \cdots f(x_n) \geq 1$$

holds for all positive numbers $x_1, x_2, \ldots, x_n$ satisfying $x_1 x_2 \ldots x_n = 1$.

16. If $x_i > 0$ for $i = 1, 2, \ldots, n$, show that,

$$\sum_{i=1}^{n} \frac{x_i^2}{x_{i+1}} \geq \sum_{i=1}^{n} x_i$$

where the indices are read modulo n.

17. If $x_i > 0$ for $i = 1, 2, \ldots, n$, show that:

$$\sum_{i=1}^{n} \frac{x_i^2}{x_i^2 + x_{i+1} x_{i+2}} \leq n - 1,$$

where the indices are read modulo n.

18. Let $a_0, a_1, \ldots, a_{n-1}$ $(n \geq 3)$ be positive integers written on the circle such that each of them divides the sum of its neighbours. Let us denote:

$$S_n = \sum_{i=1}^{n} \frac{a_{i-1} + a_{i+1}}{a_i}$$

For each n, determine the maximum and minimum possible values for S_n.

19. Let $f : [0,1] \to \mathbb{R}$ satisfy $f(0) = f(1)$ and

$$|f(a) - f(b)| \leq |a - b| \quad \text{for all } a, b \in [0,1].$$

Show that

$$|f(a) - f(b)| \leq \frac{1}{2} \quad \text{for all } a, b \in [0,1].$$

20. Let $a_1, a_2, \ldots, a_n$ be n real numbers. Show that $a_i + a_j \geq 0$ for all $i \neq j$ if and only if

$$\sum_{i=1}^{n} a_i x_i \geq \sum_{i=1}^{n} a_i x_i^2 \quad \text{for all } x_i \geq 0 \text{ such that } \sum_{i=1}^{n} x_i = 1.$$

21. Let $n \geq 3$ be an integer. Suppose that the inequality

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^2 > (n-1)\left(a_1^4 + a_2^4 + \cdots + a_n^4\right)$$

holds for some positive real numbers $a_1, a_2, \ldots, a_n$. Prove that a_i, a_j, a_k are the sides of some triangle for all i, j, k.

22. (1978 IMO Q5) Let $\{a_k\}$ be a sequence of pairwise distinct positive integers. Prove that:

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}$$

23. Let $z_1, z_2, \ldots, z_n$ be complex numbers such that $\sum_{i=1}^{n} |z_i| = 1$. Show that

$$\left| \sum_{i \in S} z_i \right| \geq \frac{1}{6}$$

for some subset S of $\{z_1, z_2, \ldots, z_n\}$.

24. Consider n complex numbers z_k such that $|z_k| \leq 1$ for $k = 1, 2, \ldots, n$. Prove that there exist $e_1, e_2, \ldots, e_n \in \{-1, 1\}$ such that for any $m \leq n$,

$$|e_1 z_1 + e_2 z_2 + \cdots + e_n z_n| \leq 2.$$

25. Find all non-negative real numbers $a_1 \leq a_2 \leq \cdots \leq a_n$ satisfying:

$$\sum_{i=1}^{n} a_i = 96, \quad \sum_{i=1}^{n} a_i^2 = 144, \quad \sum_{i=1}^{n} a_i^3 = 216.$$

26. (1999 APMO Q2) Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, \ldots$. Prove that:

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for each positive integer n.

27. Let $a_1, a_2, \ldots$ be non-negative numbers satisfying $a_{n+m} \leq a_n + a_m$ for all $m, n \in \mathbb{N}$. Prove that:

$$a_n \leq m a_1 + \left(\frac{n}{m} - 1\right) a_m$$

for all $n \geq m$.

28. (2001 IMO Shortlist) Let $a_0, a_1, \ldots$ be an arbitrary infinite sequence of positive numbers. Show that:

$$1 + a_n > a_{n-1} \sqrt[n]{2} \quad \text{for infinitely many positive integers } n.$$

29. Let $n \geq 3$, $a_i \in [2, 3]$, and $s = \sum_{i=1}^{n} a_i$. Show that:

$$\sum_{i \mod n} \frac{a_i^2 + a_{i+1}^2 - a_{i+2}^2}{a_i + a_{i+1} - a_{i+2}} \leq 2s - 2n.$$

30. Let $p(x)$ be a polynomial with real coefficients such that $p(x) > 0$ whenever $x > 0$. Show that there exists a positive integer n such that $(1 + x)^n p(x)$ has all its coefficients positive.

31. (1997 IMO Q3) Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions:

$$|x_1 + x_2 + \cdots + x_n| = 1 \quad \text{and} \quad |x_i| \leq \frac{n+1}{2} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

32. (Klamkin) Let

$$x_n = \sqrt[2]{2 + \sqrt[3]{3 + \sqrt[4]{\cdots + \sqrt[n]{n}}}}$$

Show that

$$x_{n+1} - x_n < \frac{1}{n!}.$$

33. Prove that for all positive integers n,

$$\prod_{i=1}^{n} \frac{2i-1}{2i} < \frac{1}{3n}$$

34. Prove that for natural numbers $k < l < m < n$ satisfying $kn = lm$, the following inequality holds:

$$\left(\frac{n-k}{2}\right)^2 \geq k + 2.$$