ON A SCHUR-POSITIVE FUNCTION

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Abstract. We prove Schur-positivity for a family of symmetric functions.

1. Introduction

Let e be a positive integer and let μ_e denote the set of e-th roots of 1 (in $\mathbb{C}$). For $n \in \mathbb{N}$, define the symmetric polynomial

$$W_n^{(e)}(x_1, x_2, \ldots, x_r) = \frac{1}{e^r} \sum_{\zeta_1, \ldots, \zeta_n \in \mu_e} \left(\sum_{i=1}^{r} \zeta_i \sqrt[e]{x_i} \right)^{en}.$$

Taking the inverse limit over r, these define a symmetric function $W_n^{(e)} \in \Lambda$. In terms of the monomial symmetric functions, we have the expansion

$$W_n^{(e)} = \sum_{\lambda \vdash r} \binom{er}{e\lambda} m_\lambda, \tag{1.1}$$

where $\binom{en}{e\lambda}$ is a multinomial coefficient and $e\lambda$ denotes the partition obtained from λ by multiplying all parts by e.

In [MS24, Conjecture 10.7], it was conjectured based on some computer computations that $W_n^{(2)}$ is Schur-positive. In this paper we prove

Theorem 1.1. *The symmetric function $W_n^{(e)}$ is Schur positive.*

Define coefficients a_λ by

$$W_n^{(e)} = \sum_{\lambda \vdash n} a_\lambda s_\lambda.$$

After giving a determinantal formula for a_λ, we give three proofs that $a_\lambda > 0$. The first uses the Gessel-Viennot combinatorial interpretation of a determinant of binomial coefficients. The second yields a combinatorial interpretation of the coefficients a_λ as the number of standard Young tableaux of a certain shape. The third yields a representation-theoretic interpretation of the a_λ as dimensions of simple modules over Khovanov-Lauda-Rouquier algebras (henceforth known as KLR algebras, and also known in the literature as quiver Hecke algebras), and provides a natural example of a S_n-module whose Frobenius characteristic is equal to $W_n^{(e)}$.

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2. A DETERMINANTAL FORMULA

We use standard notation for symmetric functions. When λ is a partition, ℓ can be any integer greater than or equal to the length of λ.

Define a ring homomorphism $\varphi^{(e)} : \Lambda \to \mathbb{Q}$ by

$$\varphi^{(e)}(h_n) = \frac{1}{(en)!}$$

Theorem 2.1. *The function $W_n^{(e)}$ has the Schur expansion*

$$W_n^{(e)} = (en)! \sum_{\lambda \vdash n} \varphi^{(e)}(s_\lambda) s_\lambda.$$

Proof. We prove that if $f \in \Lambda_n$, then

$$\langle W_n, f \rangle = (en)! \varphi^{(e)}(f).$$

By linearity it suffices to prove this when $f = h_\lambda$. It now follows from the fact that the complete symmetric functions are a dual basis to the monomial symmetric functions, together with the monomial decomposition (1.1). Since the Schur functions are an orthonormal basis, the theorem follows. $\square$

Corollary 2.2. *The coefficient a_λ is given by*

$$a_\lambda = (en)! \det \left(\frac{1}{(e(\lambda_i + j - i))!} \right)_{i,j=1}^{\ell} \tag{2.1}$$

Proof. This is immediate from Theorem 2.1 and the Jacobi-Trudi formula. $\square$

3. FIRST PROOF OF POSITIVITY

In the determinant (2.1), we put the i-th row over the common denominator $(e(\lambda_i + \ell - i))!$ and divide the entries in the j-th row by $(e(\ell - j))!$. This yields

$$a_\lambda = \frac{(en)! \prod_{j=1}^{\ell-1} (ej)!}{\prod_{i=1}^{\ell} (e(\lambda_i + \ell - i))!} \det \left(\binom{e(\lambda_i + \ell - i)}{e(\ell - j)} \right)_{i,j=1}^{\ell}$$

By [GV85, Theorem 1], this determinant of binomial coefficients is equal to the number of ℓ-tuples of non-intersecting lattice paths, starting at the points $(0, -e(\lambda_i + \ell - i))$ and ending at the points $(e(\ell - j), e(\ell - j))$ with all steps either east or north. In particular this number is positive, giving the first proof of Schur postivity of $W_n^{(e)}$.

4. SECOND PROOF OF POSITIVITY

Given a partition λ of n, define the partitions μ and ν by

$$\nu = (e\lambda_1 + (\ell-1)(e-1), e\lambda_2 + (\ell-2)(e-1), \cdots, e\lambda_\ell), \qquad \mu = ((\ell-1)(e-1), (\ell-2)(e-1), \cdots, 0)$$

and define the skew shape $\xi(\lambda) = \nu/\mu$. In this section, we prove

Theorem 4.1. *The coefficient a_λ is equal to the number of standard Young tableau of shape $\xi(\lambda)$.*

Proof. This is an immediate consequence of the fact [Ait43] that for any skew shape $\theta = \alpha/\beta$, we have

$$\frac{\#\{SYT(\theta)\}}{|\theta|!} = \det \left(\frac{1}{(\alpha_i + j - i - \beta_j)!} \right)_{i,j}. \tag{4.1}$$

$\square$

A combinatorial consequence of the results we have proved is

Corollary 4.2. *We have*

$$\sum_{\lambda\vdash n} \#\mathrm{SYT}(\lambda) \times \#\mathrm{SYT}(\xi(\lambda)) = \frac{(en)!}{(e!)^n}.$$

The $e = 1$ case of this result has a combinatorial proof via Robinson-Schensted, so we may ask if there is an e-Robinson-Schensted correspondence that produces a bijective proof of this identity.

5. Third proof of positivity

For reasons of space we punt the definition of KLR algebras to [McN17]. We will use KLR algebras as in that paper, in particular we work over a field of characteristic zero, and all polynomials $Q_{ij}(u,v)$ are $\pm$ a power of $u-v$. We work in type $A^{(1)}_{e-1}$ and let δ be the minimal imaginary root. Then there exists a representation L of the KLR algebra $R(\delta)$ whose character satisfies

$$\mathrm{ch}(L) = [0, 1, 2, \cdots, e-1].$$

Then [McN17, §17] constructs for each partition $\lambda \vdash n$ a simple module $L(\lambda)$ for the KLR algebra $R(n\delta)$, so that we have

$$L^{\circ n} \cong \bigoplus_{\lambda\vdash n} L(\lambda) \otimes S^\lambda$$

as $R(n\delta) - S_n$-bimodules (here S^λ are the Specht modules).

We will prove

Theorem 5.1. *For any partition λ of n, we have $a_\lambda = \dim L(\lambda)$.*

Proof. Let $\mathcal{C}_n$ be the full subcategory of finite dimensional $R(n\delta)$-modules with all Jordan-Holder quotients isomorphic to $L(\lambda)$ for some partition λ. The the results of [McN17] imply the existence of an isomorphism,

$$\bigoplus_{n=0}^{\infty} K_0(\mathcal{C}_n) \cong \Lambda.$$

under which the class of the simple module $L(\lambda)$ gets sent to s_λ.

The product on the left hand side is given by induction, which implies that $[M] \mapsto \frac{\dim M}{(en)!}$ for $M \in \mathcal{C}_n$ is a ring homomorphism.

We need to show that this ring homomorphism agrees with $\varphi^{(e)}$. Since the complete symmetric functions generate Λ, it follows from the fact that $L(1^n)$ is one-dimensional, which is easy to see as we can explicitly construct it as the module with character $[(0, 1, 2, \ldots, e-1)^n]$ □

As a consequence, we obtain

Corollary 5.2. *The Frobenius character of $L^{\circ n}$ is equal to $W^{(e)}_n$.*

We remark also that as we now know the dimensions of the modules $L(\lambda)$, we can identify them with skew-Specht modules from [MNSS25]. In particular there is a skew-Specht module $\mathbf{S}^{\zeta(\lambda)}$ constructed whose head is $L(\lambda)$ according to [MNSS25, Theorem B]. In the characteristic zero case we are considering, this surjection from $\mathbf{S}^{\zeta(\lambda)}$ to $L(\lambda)$ is an isomorphism by comparing their dimensions. (In general skew-Specht modules are not irreducible, even for purely imaginary skew-Specht modules in characteristic zero).

References

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